Part 0. Mathematics Foundations

A. Summations

B. Sets, Etc.

A. Summations

Summation formulas and properties:

Given a sequence $a_1, a_2, ..., a_n$ of numbers, the finite sum $a_1 + a_2 + ... + a_n$, where n is a nonnegative integer, can be written $\sum_{k=1}^{n} a_k$. The infinite sum $a_1 + a_2 + ...$ can be written $\sum_{k=1}^{\infty} a_k$, which is interpreted to mean $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$.

Linearity: $\sum_{k=1}^{n}(ca_k + b_k) = c\sum_{k=1}^{n} + \sum_{k=1}^{n} b_k$, where c is a constant.

The linearity property can be exploited to manipulate summations incorporating asymptotic notation:
$\sum_{k=1}^{n} \Theta(f(k)) = \Theta(\sum_{k=1}^{n} f(k))$.

Arithmetic series: $\sum_{k=1}^{n} k = 1 + 2 + ... + n = \frac{1}{2}n(n+1) = \Theta(n^2)$.
Sum of squares: $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
Sum of cubes: $\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

Geometric series: for real number $x \neq 1$, the summation $\sum_{k=0}^{n} x^k = 1 + x + x^2 + x^3 ... + x^n$ is a **geometric** or **exponential series** and has the value:

$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$.

When the summation is infinite and $0 < |x| < 1$,

we have the infinite decreasing geometric series:

$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.

Harmonic series: for positive integers n, the nth *harmonic number* is:

$$\begin{aligned} H_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} && (1) \\ &= \sum_{k=1}^{n} \frac{1}{k} && (2) \\ &= ln(n) + O(1) && (3) \end{aligned}$$

Telescoping series:

for any sequence $a_0, a_1, ..., a_n$, $\sum_{k=1}^{n}(a_k - a_{k-1}) = a_n - a_0$.

Similarly, $\sum_{k=0}^{n-1}(a_k - a_{k+1}) = a_0 - a_n$.

For example,

$$\sum_{k=1}^{n-1} \frac{1}{k(k+1)} = \sum_{k=1}^{n-1}(\frac{1}{k} - \frac{1}{k+1}) \quad (4)$$

$$= 1 - \frac{1}{n} \quad (5)$$

Bounding summations

Mathematical induction:

For example, to prove that the arithmetic series
$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.
Step 1: When $n = 1$, left = 1, right = $\frac{1}{2} * 1 * 2 = 1$.
Step 2: Make inductive assumption that it holds for n. Now we prove that it holds for n+1. We have

$$\sum_{k=1}^{n+1} = \sum_{k=1}^{n} k + (n+1) \quad (6)$$

$$= \frac{1}{2}n(n+1) + (n+1) \quad (7)$$

$$= \frac{1}{2}(n+1)(n+2) \quad (8)$$

Mathematical induction can be used to show a bound as well. In this case, you do not need to guess the exact value of a summation. For example, to prove the geometric series $\sum_{k=0}^{n} 3^k \leq c3^n$ for some constant c. Proof:
Step 1: For the initial condition $n = 0$, $\sum_{k=0}^{0} 3^k = 1 \leq c$.
Step 2: Assume that the conclusion holds for n, which means $\sum_{k=0}^{n} 3^k \leq c3^n$. Now let's prove that it also holds for $n + 1$.

$$\begin{aligned}
\sum_{k=0}^{n+1} 3^k &= \sum_{k=0}^{n} 3^k + 3^{n+1} && (9) \\
&= c3^n + 3^{n+1} && (10) \\
&= (\frac{1}{3} + \frac{1}{c})c3^{n+1} && (11) \\
&\leq c3^{n+1} && (12)
\end{aligned}$$

as long as $(\frac{1}{3} + \frac{1}{c}) \leq 1$, $c \geq \frac{3}{2}$. Thus, we conclude that $\sum_{k=0}^{n} 3^k \leq c3^n$.

Bounding the terms:

Sometimes, we can get a good upper bound on a series by bounding each term of the series. For examples: $\sum_{k=1}^{n} k \leq \sum_{k=1}^{n} n = n^2$.

In general, for a series $\sum_{k=1}^{n} a^k$, if we let $a_{max} = max_{1 \leq k \leq n} a_k$, then $\sum_{k=1}^{n} a^k \leq n a_{max}$.

However, bounding each term by the largest term is a weak method when the series can in fact be bounded by a geometric series.

Give the series: $\sum_{k=0}^{n} a_k$. Suppose $a_{k+1} \leq ra_k$ for all $k \geq 0$ and r is a constant, and $0 < r < 1$. So
$a_k \leq ra_{k-1} \leq r^2 a_{k-2} \leq r^3 a_{k-3} \leq ... \leq r^k a_0$. Thus,

$$\sum_{k=0}^{n} a_k \leq \sum_{k=0}^{\infty} a_0 r^k \tag{13}$$

$$= a_0 \sum_{k=0}^{\infty} r^k \tag{14}$$

$$= a_0 \frac{1}{1-r} \tag{15}$$

To apply the above method to this example: $\sum_{k=1}^{\infty} \frac{k}{3^k}$. We rewrite it as $\sum_{k=0}^{\infty} \frac{k+1}{3^{k+1}}$. So $a_0 = \frac{1}{3}$. The ratio r is

$$\begin{aligned} r &= \frac{(k+2)/3^{k+2}}{(k+1)/3^{k+1}} && (16) \\ &= \frac{1}{3}\frac{k+2}{k+1} && (17) \\ &\leq \frac{2}{3} && (18) \end{aligned}$$

for all $k \geq 0$. Thus, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{k}{3^k} &= \sum_{k=0}^{\infty} \frac{k+1}{3^{k+1}} && (19) \\ &= \frac{1}{3}\frac{1}{1-\frac{2}{3}} && (20) \\ &= 1 && (21) \end{aligned}$$

Splitting summation: a way to obtain bounds on a difficult summation by partitioning the range of the index and then to bound each of the resulting series. For examples,

$$\begin{aligned}
\sum_{k=1}^{n} k &= \sum_{k=1}^{n/2} k + \sum_{k=n/2+1}^{n} k && (22)\\
&\geq \sum_{k=1}^{n/2} 0 + \sum_{k=n/2+1}^{n} \frac{n}{2} && (23)\\
&= (n/2)^2 && (24)\\
&= \Omega(n^2) && (25)
\end{aligned}$$

For any constant k_0, we have

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{k_0-1} a_k + \sum_{k=k_0}^{n} a_k \quad = \quad \Theta(1) + \sum_{k=k_0}^{n} a_k \tag{26}$$

Now compute the summation $\sum_{k=0}^{\infty} \frac{k^2}{2^k}$. The ratio of consecutive terms is

$$\frac{(k+1)^2/2^{k+1}}{k^2/2^k} \quad = \quad \frac{(k+1)^2}{2k^2} \tag{27}$$

$$= \quad \frac{k^2+2k+1}{2k^2} \tag{28}$$

$$= \quad \frac{1}{2} + \frac{1}{k} + \frac{1}{2k^2} \tag{29}$$

$$\leq \quad \frac{8}{9} \tag{30}$$

if $k \geq 3$. Thus, the summation can be split into

$$\sum_{k=0}^{\infty} \frac{k^2}{2^k} = \sum_{k=0}^{2} \frac{k^2}{2^k} + \sum_{k=3}^{\infty} \frac{k^2}{2^k} \tag{31}$$

$$\leq \sum_{k=0}^{2} \frac{k^2}{2^k} + \frac{9}{8} \sum_{k=3}^{\infty} (\frac{8}{9})^k \tag{32}$$

$$\leq \sum_{k=0}^{2} \frac{k^2}{2^k} + \frac{9}{8} \sum_{k=0}^{\infty} (\frac{8}{9})^k \tag{33}$$

$$= O(1) \tag{34}$$

B. Sets, etc.

A **set** is a collection of distinguishable objects, called its **members** or **elements**.

Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$
Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$
Difference: $A - B = \{x : x \in A \text{ and } x \notin B\}$

Empty set laws: $A \cap \emptyset = \emptyset$, $A \cup \emptyset = A$

Idempotency laws: $A \cap A = A$, $A \cup A = A$

Commutative laws: $A \cap B = B \cap A$, $A \cup B = B \cup A$

Associative laws: $A \cap (B \cap C) = (A \cap B) \cap C$,
$A \cup (B \cup C) = (A \cup B) \cup C$

Distributive laws: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Absorption laws: $A \cap (A \cup B) = A$, $A \cup (A \cap B) = A$

DeMorgan's laws: $A - (B \cap C) = (A - B) \cup (A - C)$,
$A - (B \cup C) = (A - B) \cap (A - C)$

Complement: consider all the sets are subsets of some larger set U called the **Universe**, the complement of set A is $U - A$.
We have $A \cap \bar{A} = \emptyset$
$A \cup \bar{A} = U$
$\bar{\bar{A}} = A$
DeMorgan's law can be rewritten with complements for any two sets $B, C \subseteq U$, we have $\overline{B \cap C} = \bar{B} \cup \bar{C}$ and $\overline{B \cup C} = \bar{B} \cap \bar{C}$

Cardinality: the number of elements in a set;
Same cardinality: two sets have the same cardinality if their elements can be put into a one-to-one correspondence;
Finite set: the cardinality of the set is a natural number. Otherwise, it is **infinite**;
Countably infinite set: An infinite set, and it can be put into a one-to-one correspondence with the natural numbers N. Otherwise, it is **uncountable**.

n-set: A finite set of n elements;

Singleton: A 1-set;

k-subset: A subset of k elements of a set;

Power set: The set of all subsets of a set, i.e., $2^{\{a,b\}} = \emptyset, \{a\}, \{b\}, \{a, b\}$;

Ordered pair: An ordered pair of two elements a and b is denoted (a, b);

Cartesian product of two sets: the set of all ordered pairs, i.e., $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$, $|A \times B| = |A| \times |B|$;

Binary relation R: a subset of the Cartesian product $A \times B$.

Properties:

(1) Reflexive: if a R a for all $a \in A$. E.g, $=$ and $\leq$, ...

(2) Symmetric: if a R b implies b R a. E.g. $=$.

(3) Transitive: if a R b and b R c implies a R c. E.g. $=$, $<$, $\leq$, ...

(4) Antisymmetric: a R b and b R a implies $a = b$. E.g. $=$.